MATHEMATICS

TRANSLATION INVARIANT MAPS $L^\infty(G) \to L^p(G)$

BY

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For a group G, a function f on G and an element x of G define the function f_x by

$$f_x(y) = f(xy) \quad (y \in G).$$

Let G be a locally compact group. Let E, F be spaces of type $L^p(G)$ $(1 \leqslant p \leqslant \infty)$ or $C_0(G)$. A map $T: E \to F$ is said to be *left invariant* if $T(f_x) = (Tf)_x$ for every $f \in E$ and $x \in G$. As is well known, under convolution $L^p(G)$ and $C_0(G)$ are modules over the Banach algebra $L^1(G)$. A *multiplier* $E \to F$ is a continuous linear map $E \to F$ that is a module homomorphism (i.e. $T(\varphi \star f) = \varphi \star Tf$ for all $\varphi \in L^1(G)$, $f \in E$). It is known that every multiplier is left invariant, but left invariant continuous linear maps $E \to F$ may fail to be multipliers in case $E = L^\infty(G)$.

It is reasonable to ask for a description of the left invariant continuous linear maps and the multipliers $E \to F$. In this field much work has been done; for more information we refer the reader to [5]. In this paper we only ask if, for certain choices of E and F, there exist non-zero left invariant maps or multipliers $E \to F$. For compact G the answer is easily seen to be affirmative. Further, the case $E = F$ is trivial, as is the case $E = C_0(G)$, $F = L^\infty(G)$. It is not hard to show that non-zero multipliers $E \to F$ exist in the cases

$$E = L^p(G), \ F = L^q(G), \ 1 \leqslant p < q \leqslant \infty,$$
$$E = L^p(G), \ F = C_0(G), \ 1 \leqslant p < \infty.$$

(See our Lemma 1.) On the other hand, in the cases

$$\left.\begin{array}{l} E = L^p(G), \ F = L^q(G), \ q < p < \infty, \\ E = C_0(G), \ F = L^q(G), \ q < \infty, \end{array}\right\} \ (G \text{ not compact})$$

no non-zero left invariant continuous linear map $E \to F$ exists. (This was proved by L. HÖRMANDER [4] for $G = \mathbb{R}^n$, but the proof is easy to generalize.) This leaves open the cases

$$E = L^\infty(G), \ F = L^p(G), \ 1 \leqslant p < \infty,$$
$$E = L^\infty(G), \ F = C_0(G).$$

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We propose to prove the following. *If G is not compact, then for $1 \leqslant p < 2$ there doesnot exist any non-zero left invariant continuous linear map $L^\infty(G) \to L^p(G)$. On the other hand, if $p \geqslant 2$ there exist non-zero multipliers $L^\infty(G) \to L^p(G)$ and $L^\infty(G) \to C_0(G)$, provided that G is either abelian or discrete amenable.*

First we reconsider the cases $L^p(G) \to L^q(G)$ $(1 \leqslant p < q \leqslant \infty)$ and $L^p(G) \to C_0(G)$ $(1 \leqslant p < \infty)$ mentioned above. Our Lemma 1 trivially implies the existence of non-zero multipliers.

We introduce the following notations. G is an abstract group or a locally compact group: its identity element is denoted by 1. In case G is locally compact we select a left Haar measure on G. The integral of a function f relative to this measure is indicated by $\int f$ or $\int f(x)dx$. The characteristic function of a set A is ξ_A.

LEMMA 1. *Let G be a locally compact group. Let $1 \leqslant p < \infty$ and let $0 \neq g \in L^p(G)$. Then there exist multipliers $T: L^p(G) \to L^q(G)$ $(p \leqslant q \leqslant \infty)$ and $T: L^p(G) \to C_0(G)$ such that $Tg \neq 0$.*

PROOF. Let $s = \dfrac{pq}{pq+p-q}$; then $1 \leqslant s < \infty$ and $\dfrac{1}{p} + \dfrac{1}{s} - \dfrac{1}{q} = 1$. Let ω be a continuous function on G that has compact support and such that $\omega(x) = \omega(x^{-1})$ for all $x \in G$. For every $f \in L^p(G)$ the function $T_\omega f$ defined by

$$(T_\omega f)(x) = \int f_x \omega \quad (x \in G)$$

is an element of $L^q(G)$ $(p \leqslant q \leqslant \infty)$ and of $C_0(G)$, while

$$\|T_\omega f\|_q \leqslant \|f\|_p \|\omega\|_s \quad (f \in L^p(G)).$$

(See the proof of [3; 20.18].) Thus, T_ω is a continuous linear map $L^p(G) \to L^q(G)$ $(p \leqslant q \leqslant \infty)$ and $L^p(G) \to C_0(G)$. It is easy to show that this T_ω actually is a multiplier. It follows from [3; 20.15] that ω can be chosen so that $T_\omega g \neq 0$.

We proceed to prove that for non-compact G and for $p < 2$ there exist no non-zero left invariant continuous linear maps $L^\infty(G) \to L^p(G)$. (Theorem 3.)

LEMMA 2. *Let $N \in \mathbf{N}$ and let $\mu_0, \mu_1, \ldots, \mu_{2N}$ be real signed measures on a σ-algebra $\mathscr{B}$ of subsets of a set X such that $\|\mu_i\| = 1$ for each i. Then there exist $t_0, t_1, \ldots, t_{2N} \in \{-1, +1\}$ such that $\|\sum t_i \mu_i\| \geqslant 2^{-2N}(2N+1)\binom{2N}{N}$.*

N.B. 1. The content of the lemma may become clearer if one observes that asymptotically $2^{-2N}(2N+1)\binom{2N}{N} \sim 2\pi^{-\frac{1}{2}}N^{\frac{1}{2}}$.

N.B. 2. If for $\mathscr{B}$ one takes the Borel σ-algebra of $[0, 1]$ and for μ_i the measure induced by the Lebesgue measure and the i-th Rademacher function, then for all $t_0, \ldots, t_{2N}$ we have exactly $\|\sum t_i \mu_i\| = 2^{-2N}(2N+1)\binom{2N}{N}$.

(A proof of this fact is hidden in the proof of the lemma.) In this sense the lemma cannot be strengthened.

PROOF. By S we denote the set of all 2^{2N+1} sequences $s=(s_0, s_1, \ldots, s_{2N})$ where $s_i \in \{-1, +1\}$ for each i. In a natural way S is a group. Let $m=\sum |\mu_i|$ and let h_i be $\mathscr{B}$-measurable such that $\mu_i = h_i m$. For each $s \in S$ let

$$X_s = \{x \in X : h_i(x) \geqslant 0 \text{ if } s_i = 1;\ h_i(x) < 0 \text{ if } s_i = -1\}.$$

The sets X_s are pairwise disjoint, their union is X and

$$s_i \mu_i \geqslant 0 \text{ on } X_s \quad (s \in S;\ i = 0, 1, \ldots, 2N).$$

For $u \in S$ we put $\operatorname{sgn} u = +1$ if in the sequence $(u_0, u_1, \ldots, u_{2N})$ the number $+1$ occurs more often than -1; otherwise, $\operatorname{sgn} u = -1$.

If $s, t \in S$, then

$$\Big(\sum_i t_i \mu_i\Big)(X_s) = \sum_i s_i t_i |\mu_i|(X_s),$$

so that

$$\Big|\Big(\sum_i t_i \mu_i\Big)(X_s)\Big| \geqslant \sum_i s_i t_i (\operatorname{sgn} st) |\mu_i|(X_s) \quad (s, t \in S).$$

Consequently,

$$\sum_t \Big\|\sum_i t_i \mu_i\Big\| \geqslant \sum_t \sum_s \Big|\Big(\sum_t t_i \mu_i\Big)(X_s)\Big| \geqslant$$

$$\geqslant \sum_{s,t,i} s_i t_i (\operatorname{sgn} st)|\mu_i|(X_s) = \sum_{s,i} \Big\{\sum_t s_i t_i (\operatorname{sgn} st)\Big\} |\mu_i|(X_s).$$

Now $\sum_t s_i t_i (\operatorname{sgn} st)$ does not depend on s or i. If for the moment we denote this number by c_N, then we obtain

$$\sum_t \Big\|\sum_i t_i \mu_i\Big\| \geqslant c_N \sum_i \sum_s |\mu_i|(X_s) = c_N \sum_i \|\mu_i\| = c_N(2N+1).$$

Therefore, there must be a $t \in S$ for which $\|\sum_i t_i \mu_i\| \geqslant 2^{-2N-1} c_N (2N+1)$. It remains to prove that $c_N = 2\binom{2N}{N}$. This is not difficult to do. First, observe that $c_N = \sum_t t_0(\operatorname{sgn} t)$. The number of elements t of S for which $t_0 = \operatorname{sgn} t = +1$ equals the number of subsets of $\{1, \ldots, 2N\}$ of at least N elements, which is $\frac{1}{2}\left(2^{2N} + \binom{2N}{N}\right)$. Hence, there are exactly $2^{2N} + \binom{2N}{N}$ elements t of S for which $t_0 \operatorname{sgn} t = 1$. For the other $2^{2N+1} - \left(2^{2N} + \binom{2N}{N}\right)$ elements t of S we have $t_0 \operatorname{sgn} t = -1$. Thus,

$$c_N = 2^{2N} + \binom{2N}{N} - \left(2^{2N+1} - \left[2^{2N} + \binom{2N}{N}\right]\right) = 2\binom{2N}{N}.$$

THEOREM 3. *Let G be a non-compact locally compact group. If $1 \leqslant p < 2$, then the only left invariant continuous linear map $L^\infty(G) \to L^p(G)$ is* 0.

PROOF. We may restrict ourselves to the scalar field $\mathbf{R}$. Let $1 \leqslant p < \infty$ and let T be a non-zero left invariant continuous linear map $L^\infty(G) \to L^p(G)$. We prove that $p \geqslant 2$.

Let m be a left Haar measure on G. Take a compact set $A \subset G$ so that $\int_A Tf \neq 0$ for some $f \in L^\infty(G)$. As G is not compact there exist $x_0 = 1$, $x_1, x_2, \ldots \in G$ such that the sets $x_i A$ are pairwise disjoint. For $i = 0, 1, 2, \ldots$ define $\varphi_i \in L^\infty(G)^*$ by

$$\varphi_i(f) = \int_{x_i A} Tf.$$

We have $\varphi_i(f) = \varphi_0(f_{x_i})$, so that $\|\varphi_i\| = \|\varphi_0\|$ for every i. By our choice of A, $\varphi_0 \neq 0$. Thus, we may assume

$$\|\varphi_i\| = 1 \quad (i = 0, 1, 2, \ldots).$$

Let $q = \dfrac{p}{p-1}$. By Hölder's Inequality, for all $f \in L^\infty(G)$ and $i \in \mathbf{N}$,

$$|\varphi_i(f)|^p = |\int_{x_i A} Tf|^p \leqslant \|(Tf)\,\xi_{x_i A}\|_p{}^p \|\xi_{x_i A}\|_q{}^p =$$

$$= (\int_{x_i A} |Tf|^p)\, m(A)^{p-1}.$$

Hence,

$$\sum_i |\varphi_i(f)|^p \leqslant \|Tf\|^p m(A)^{p-1} \quad (f \in L^\infty(G)).$$

Setting

$$B = \{f \in L^\infty(G) : \|f\| \leqslant 1\}$$

we have

$$\sum_i |\varphi_i(f)|^p \leqslant \|T\|^p m(A)^{p-1} \quad (f \in B). \tag{1}$$

As is well-known, for some compact space X there exists a positive linear isometry of $L^\infty(G)$ onto $C(X)$. By the Riesz Representation Theorem the elements of $L^\infty(G)^*$ correspond to the signed Radon measures on the Borel σ-algebra of X. This observation enables us to make use of the lemma: we see now that for every $N \in \mathbf{N}$ there exist $t_0, t_1, \ldots, t_{2N} \in \{-1, +1\}$ such that $\|\sum_{i=0}^{2N} t_i \varphi_i\| \geqslant 2^{-2N}(2N+1)\binom{2N}{N}$, i.e.

$$\sup_{f \in B} |\sum_{i=0}^{2N} t_i \varphi_i(f)| \geqslant 2^{-2N}(2N+1)\binom{2N}{N}. \tag{2}$$

Another application of Hölder's Inequality yields

$$\left\{\begin{aligned} \sup_{f \in B} |\sum_{i=0}^{2N} t_i \varphi_i(f)| &\leqslant \sup_{f \in B} (\sum_{i=0}^{2N} |t_i|^q)^{1/q} (\sum_{i=0}^{2N} |\varphi_i(f)|^p)^{1/p} = \\ &= (2N+1)^{1/q} \sup_{f \in B} (\sum_{i=0}^{2N} |\varphi_i(f)|^p)^{1/p}. \end{aligned}\right. \tag{3}$$

By combining (1), (2) and (3) we obtain

$$2^{-2N}(2N+1)\binom{2N}{N} \leqslant (2N+1)^{1/q} \|T\| m(A)^{1-1/p} \quad (N \in \mathbf{N}),$$

so that

$$\sup_N 2^{-2N}(2N+1)^{1/p}\binom{2N}{N}<\infty.$$

But

$$2^{-2N}(2N+1)^{1/p}\binom{2N}{N}\sim 2^{1/p}\pi^{-\frac{1}{2}}N^{\frac{1}{p}-\frac{1}{2}}\ (N\to\infty).$$

It follows that $p\geqslant 2$.

We are now going to consider multipliers $L^\infty(G)\to C_0(G)$ and $L^\infty(G)\to$ $\to L^p(G)$ where $p\geqslant 2$. The following lemma implies that it suffices to consider maps $C_{ru}(G)\to L^2(G)$. Here $C_{ru}(G)$ is the space of all bounded right uniformly continuous functions on G. $C_{ru}(G)$ is a left invariant closed linear subspace of $L^\infty(G)$. By [3; 32.45(b)], $C_{ru}(G)=\{\varphi\star f\colon \varphi\in L^1(G);$ $f\in L^\infty(G)\}$.

LEMMA 4. *Let G be a locally compact group. Assume that there exists a non-zero left invariant continuous linear map $S\colon C_{ru}(G)\to L^2(G)$. Then there exist non-zero multipliers $L^\infty(G)\to L^p(G)$ $(2\leqslant p\leqslant\infty)$ and $L^\infty(G)\to C_0(G)$.*

PROOF. By Lemma 1 we are done if we can construct a non-zero multiplier $L^\infty(G)\to L^2(G)$.

Let $f\in L^\infty(G)$. We know [3; 20.16] that $\varphi\star f\in C_{ru}(G)$ for all $\varphi\in L^1(G)$. It follows that $\varphi\mapsto S(\varphi\star f)$ is a left invariant continuous linear map $L^1(G)\to L^2(G)$. By [2; 3.11] there exists a unique $Tf\in L^2(G)$ such that

$$S(\varphi\star f)=\varphi\star Tf\ \ (\varphi\in L^1(G)). \tag{1}$$

Moreover, $\|Tf\|$ is equal to the norm of the map $\varphi\mapsto S(\varphi\star f)$, i.e. $\|Tf\|\leqslant$ $\leqslant\|S\|\,\|f\|$.

We see that (1) defines a continuous linear map $T\colon L^\infty(G)\to L^2(G)$. If $\psi\in L^1(G)$ and $f\in L^\infty(G)$ then

$$S(\varphi\star(\psi\star f))=S((\varphi\star\psi)\star f)=(\varphi\star\psi)\star Tf=\varphi\star(\psi\star Tf)\ \ (\varphi\in L^1(G)),$$

so that

$$T(\psi\star f)=\psi\star Tf\ \ (\psi\in L^1(G);\ f\in L^\infty(G)) \tag{2}$$

and T is a multiplier. In particular, from (1) and (2) we see that $S=T$ on the subset $\{\psi\star f\colon \psi\in L^1(G);\ f\in L^\infty(G)\}$ of $L^\infty(G)$, i.e. on $C_{ru}(G)$. Hence, T is an extension of S and $T\neq 0$.

Let G be a group. A linear subspace D of $l^\infty(G)$ is said to be *left invariant* if $f_x\in D$ for all $f\in D$ and all $x\in G$. For such a D, an element φ of D^* is *left invariant* if $\varphi(f_x)=\varphi(f)$ for all $f\in D$ and all $x\in G$.

Now let D be a closed linear subspace of $l^\infty(G)$ containing the constant functions. (We do not require D to be left invariant.) A *mean* on D is an element M of D^* for which $M(1)=1$ and $\|M\|=1$. If M is a mean on $l^\infty(G)$ then $M(f)\geqslant 0$ if $f\in l^\infty(G)$ is real-valued and $f\geqslant 0$. G is called *amenable* if there exists a left invariant mean on $l^\infty(G)$. A topological group G is *amenable* if there exists a left invariant mean on the subspace of $l^\infty(G)$ that consists of the bounded continuous functions on G.

EXTENSION LEMMA 5. *Let G be an amenable group, D a left invariant closed linear subspace of $l^\infty(G)$. Then every left invariant element of D^* extends to a left invariant element of $l^\infty(G)^*$ of the same norm.*

PROOF. Let $H \subset l^\infty(G)$ be the closed linear span of $\{f-f_x\colon f \in l^\infty(G);\ x \in G\}$ and let $H_D \subset D$ be the closed linear span of $\{f-f_x\colon f \in D;\ x \in G\}$. It follows from [1; 2.14] that

$$\text{dist}\,(g, H) = \text{dist}\,(g, H_D) \quad (g \in D), \tag{1}$$

(so that $H_D = H \cap D$). Let $\varphi \in D^*$ be left invariant. Then φ vanishes on H_D. Hence, for all $g \in D$ we find

$$|\varphi(g)| = \inf\{|\varphi(g+h)| : h \in H_D\} \leqslant \inf\{\|\varphi\|\,\|g+h\| : h \in H_D\}.$$

Thus,

$$|\varphi(g)| \leqslant \|\varphi\|\,\text{dist}\,(g, H_D) \quad (g \in D). \tag{2}$$

Combining (1) and (2) one obtains

$$|\varphi(g)| \leqslant \|\varphi\|\,\|g+h\| \quad (g \in D;\ h \in H).$$

It follows that the formula

$$g+h \mapsto \varphi(g) \quad (g \in D;\ h \in H)$$

defines a function φ_1 on $D+H$. Moreover, φ_1 is linear and $\|\varphi_1\| \leqslant \|\varphi\|$. By the Hahn-Banach Theorem, φ_1 extends to a $\varphi_2 \in l^\infty(G)^*$ for which $\|\varphi_2\| = \|\varphi_1\| \leqslant \|\varphi\|$. This φ_2 is an extension of φ, so that $\|\varphi_2\| = \|\varphi\|$. Further, $\varphi_2 = 0$ on H. Consequently, φ_2 is left invariant.

LEMMA 6. *Let G be an infinite amenable discrete group. There exist a left invariant mean M on $l^\infty(G)$ and a real-valued $j \in l^\infty(G)$ such that $M(j_x j_y) = \delta_{xy}$ $(x, y \in G)$, where δ is the Kronecker symbol ($\delta_{xy} = 0$ if $x \neq y$, $\delta_{xy} = 1$ if $x = y$).*

PROOF. The cardinal number of a set X is denoted $\# X$. Let $\mathcal{M} = \# G$ and let μ be the first ordinal number whose cardinality is $\mathcal{M}$. If $\mathcal{F}$ is the set of all finite subsets of G, then the cardinal number of $\mathbf{N} \times \mathcal{F}$ is just $\mathcal{M}$. If $X \in \mathcal{F}$, if $\beta < \mu$ and if for every $\alpha < \beta$, X_α is a finite subset of G, then $\# \bigcup_{\alpha<\beta} X^{-1} X_\alpha < \mathcal{M}$ so that there exists an $x \in G$, $x \notin \bigcup_{\alpha<\beta} X^{-1} X_\alpha$. For such an x we have $Xx \cap X_\alpha = \emptyset$ $(\alpha < \beta)$. It follows by transfinite induction that *there exists a family $(x_{n,X})_{n \in \mathbf{N}, X \in \mathcal{F}}$ of elements of G such that*

$$Xx_{n,X} \cap Yx_{m,Y} = \emptyset \text{ if } n \neq m \text{ or } X \neq Y.$$

For $X \in \mathcal{F}$ let S_X denote the set of all maps of X into the two-element set $\{-1, 1\}$. From the above we infer the existence of a $j\colon G \to \{-1, 1\}$

with the following property: *For any* $X \in \mathscr{F}$ *and* $s \in S_X$ *there exists a* $z \in G$ *such that* $j(xz) = s(x)$ $(x \in X)$.

For all $X \in \mathscr{F}$ and $s \in S_X$ put

$$A_s = \{z \in G : s(x) = j(xz) \text{ for all } x \in X\}.$$

We have just seen that $A_s \neq \emptyset$. We observe that, if $X \in \mathscr{F}$, then the A_s $(s \in S_X)$ form a partition of G by $2^{\#X}$ non-empty sets. Moreover, if $Y \in \mathscr{F}$ and $Y \supset X$, then every A_s $(s \in S_X)$ is a union of exactly $2^{\#Y - \#X}$ sets of the form A_t where $t \in S_Y$. It follows that there exists a unique linear function N on the linear hull D of $\{\xi_{A_s} : s \in S_X; X \in \mathscr{F}\}$ for which

$$N(\xi_{A_s}) = 2^{-\#X} \quad (X \in \mathscr{F};\ s \in S_X).$$

Clearly, N is a mean on D. One easily sees that N is left invariant. By Lemma 5, N extends to a left invariant mean M on $l^\infty(G)$.

For every $x \in G$, $M(j_x j_x) = M(1) = N(1) = 1$. Now let $x, y \in G$ be distinct. We show $M(j_x j_y) = 0$. Take $X = \{x, y\}$. The set S_X consists of four elements s_{11}, $s_{1,-1}$, $s_{-1,1}$, $s_{-1,-1}$ where $s_{ij}(x) = i$, $s_{ij}(y) = j$ $(i, j \in \{-1, 1\})$. We have

$$j_x j_y = \xi_{A_{s_{11}}} - \xi_{A_{s_{1,-1}}} - \xi_{A_{s_{-1,1}}} + \xi_{A_{s_{-1,-1}}}$$

so that indeed $M(j_x j_y) = N(j_x j_y) = 0$.

Corollary 7. *Let G be an amenable discrete group. Then there exists a non-zero left invariant continuous linear map $l^\infty(G) \to l^2(G)$. Therefore there also exist non-zero left invariant continuous linear maps $l^\infty(G) \to l^p(G)$ $(p \geqslant 2)$ and $l^\infty(G) \to c_0(G)$.*

Proof. We may assume G to be infinite. Let M, j be as above. The formula

$$\langle f, g \rangle = M(f\bar{g})$$

defines a semi-inner product $\langle\ ,\ \rangle$ on $l^\infty(G)$. By the preceding lemma, $\langle j_x, j_y \rangle = \delta_{xy}$ $(x, y \in G)$. Therefore, if $f \in l^\infty(G)$, then $\sum_{x \in G} |\langle f, j_x \rangle|^2 \leqslant \langle f, f \rangle = M(|f|^2) \leqslant \|f\|^2$. Thus, we can define a continuous linear map $T : l^\infty(G) \to l^2(G)$ by

$$(Tf)(x) = \langle f, j_{x^{-1}} \rangle \quad (f \in l^\infty(G);\ x \in G).$$

We have $(Tj)(1) = \langle j, j \rangle = 1$, so $T \neq 0$. Further, T is left invariant since for all $f \in l^\infty(G)$ and $x, y \in G$,

$$\begin{aligned}(T(f_y))(x) &= M(f_y j_{x^{-1}}) = M((f_y j_{x^{-1}})_{y^{-1}}) = M(f j_{x^{-1}y^{-1}}) = \\ &= M(f j_{(yx)^{-1}}) = (Tf)(yx) = (Tf)_y(x).\end{aligned}$$

Theorem 8. *If G is an amenable locally compact group that has a compact open normal subgroup, then there exists a non-zero left invariant continuous linear map $C_{ru}(G) \to L^2(G)$.*

PROOF. Let H be a compact open normal subgroup of G. Then G/H is amenable and discrete. By Corollary 7, there exists a non-zero left invariant continuous linear map $S: l^\infty(G/H) \to l^2(G/H)$. For $x \in G$ we denote by $\bar{x}$ the corresponding element of G/H. Every $f \in C_{ru}(G)$ determines an $\bar{f} \in l^\infty(G/H)$ by

$$\bar{f}(\bar{x}) = \int_H f(xy)\, dm_H(y)$$

where m_H is the normalized Haar measure on H. $f \mapsto \bar{f}$ is a continuous linear surjection $C_{ru}(G) \to l^\infty(G/H)$ and

$$(f_x)^- = (\bar{f})_{\bar{x}} \quad (f \in C_{ru}(G);\ x \in G).$$

The formula

$$(Tf)(x) = (S\bar{f})(\bar{x}) \quad (f \in C_{ru}(G);\ x \in G)$$

now defines a map $T: C_{ru}(G) \to L^2(G)$ which is easily seen to satisfy the requirements.

For a group G and a Hilbert space D we denote by $l^\infty{}_D(G)$ the Banach space of all bounded maps $G \to D$, provided with the sup-norm $\|f\| = \sup\{\|f(x)\|: x \in G\}$. For $f \in l^\infty(G)$ and $\zeta \in D$ define $f \otimes \zeta: G \to D$ by

$$(f \otimes \zeta)(x) = f(x)\,\zeta.$$

Then $f \otimes \zeta \in l^\infty{}_D(G)$ and $\|f \otimes \zeta\| = \|f\|\ \|\zeta\|$. By $l^2{}_D(G)$ we indicate the space of all $f \in l^\infty{}_D(G)$ for which $\sum_{x \in G} \|f(x)\|^2$ is finite; this $l^2{}_D(G)$ is a Hilbert space under the inner product

$$\langle f, g\rangle = \sum_{x \in G} \langle f(x), g(x)\rangle \quad (f, g \in l^2{}_D(G)).$$

LEMMA 9. *Let G be an amenable discrete group, let M, j be as in Lemma 6. Let D be a Hilbert space. Then there exists a continuous linear $T: l^\infty{}_D(G) \to l^2{}_D(G)$ with the following properties.*

(a) $T(f_x) = (Tf)_x$ $(f \in l^\infty{}_D(G);\ x \in G)$.
(b) $T(A \circ f) = A \circ Tf$ $(f \in l^\infty{}_D(G);\ A: D \to D$ *continuous linear*$)$.
(c) $[T(j \otimes \zeta)](1) = \zeta$ $(\zeta \in D)$.

PROOF. In $l^\infty{}_D(G)$ we introduce a semi-inner product $\langle\ ,\ \rangle_0$ by

$$\langle f, g\rangle_0 = M(\langle f(\,.\,), g(\,.\,)\rangle) \quad (f, g \in l^\infty{}_D(G)).$$

The corresponding seminorm we denote by $\|\ \|_0$. We write $f \perp g$ if $\langle f, g\rangle_0 = 0$.

For $x, y \in G$ and $\zeta, \eta \in D$,

$$\langle j_x \otimes \zeta,\ j_y \otimes \eta\rangle_0 = \langle \zeta, \eta\rangle \delta_{xy}.$$

In particular, the restriction of $\langle\ ,\ \rangle_0$ to the linear subspace $D_x = \{j_x \otimes \zeta: \zeta \in D\}$ of $l^\infty{}_D(G)$ is a true inner product, inducing a norm. Relative to this norm, $\zeta \mapsto j_x \otimes \zeta$ is an isometry of D onto D_x. It follows that for every $f \in l^\infty{}_D(G)$ and $x \in G$ there exists a unique $T_x f \in D$ such that

$$\langle j_x \otimes \zeta, f\rangle_0 = \langle \zeta, T_x f\rangle \quad (\zeta \in D).$$

We have

(1) $$D_x \perp D_y \quad (x, y \in G;\ x \neq y);$$

(2) $$j_x \otimes T_x f \in D_x \quad (x \in G;\ f \in l^\infty{}_D(G));$$

(3) $$(f - j_x \otimes T_x f) \perp D_x \quad (x \in G;\ f \in l^\infty{}_D(G)).$$

It follows that

$$\sum_{x \in G} \|j_x \otimes T_x f\|_0{}^2 \leqslant \|f\|_0{}^2 \quad (f \in l^\infty{}_D(G)).$$

Now $\|j_x \otimes T_x f\|_0 = \|T_x f\|$, and $\|f\|_0{}^2 = M(\|f(\,.\,)\|^2) \leqslant \|f\|^2 \quad (f \in l^\infty{}_D(G))$. Thus, the formula

$$(Tf)(x) = T_{x^{-1}} f \quad (f \in l^\infty{}_D(G);\ x \in G)$$

defines a continuous linear map $T : l^\infty{}_D(G) \to l^2{}_D(G)$.

It is now easy to see that this T has the properties (a), (b) and (c).

Lemma 10. *Let G be a locally compact group that admits a non-zero left invariant continuous linear map $S : C_{ru}(G) \to L^2(G)$. Then there exist non-zero left invariant continuous linear maps $C_{ru}(\mathbf{Z} \times G) \to L^2(\mathbf{Z} \times G)$ and $C_{ru}(\mathbf{R} \times G) \to L^2(\mathbf{R} \times G)$.*

N.B. Instead of $\mathbf{Z}$ we could have taken any discrete amenable group.

Proof. For $f : \mathbf{Z} \times G \to \mathbf{C}$ and $n \in \mathbf{Z}$ define $f^{(n)} : G \to \mathbf{C}$ by

$$f^{(n)}(x) = f(n, x) \quad (x \in G).$$

Set $D = L^2(G)$. The formula

$$(Vf)(n) = S(f^{(n)}) \quad (f \in C_{ru}(\mathbf{Z} \times G);\ n \in \mathbf{Z})$$

defines a continuous linear map $V : C_{ru}(\mathbf{Z} \times G) \to l^\infty{}_D(\mathbf{Z})$. By Lemma 9 we have a continuous linear map $T : l^\infty{}_D(\mathbf{Z}) \to l^2{}_D(\mathbf{Z})$ for which

(a) $T(h_m) = (Th)_m \quad (h \in l^\infty{}_D(\mathbf{Z});\ m \in \mathbf{Z})$;
(b) $T(A \circ h) = A \circ Th \quad (h \in l^\infty{}_D(\mathbf{Z});\ A : D \to D$ continuous linear$)$;
(c) $[T(j \otimes \zeta)](0) = \zeta \quad (\zeta \in D)$

where j is a suitable element of $l^\infty(\mathbf{Z})$.

Then TV is continuous linear $C_{ru}(\mathbf{Z} \times G) \to l^2{}_D(\mathbf{Z})$. But there exists a natural isometry U of $l^2{}_D(\mathbf{Z})$ onto $L^2(\mathbf{Z} \times G)$ given by

$$(Uh)(n, x) = h(n)(x) \quad (h \in l^2{}_D(\mathbf{Z});\ n \in \mathbf{Z};\ x \in G).$$

Thus we obtain a continuous linear map $W=UTV$ of $C_{ru}(\mathbf{Z}\times G)$ into $L^2(\mathbf{Z}\times G)$ such that

$$(Wf)^{(n)}=(TVf)(n) \quad (f\in C_{ru}(\mathbf{Z}\times G);\ n\in\mathbf{Z}).$$

It is extremely tedious but not difficult to prove that W is left invariant. To prove that $W\neq 0$, observe that there exists a $g\in C_{ru}(G)$ for which $Sg\neq 0$. Define $f\in C_{ru}(\mathbf{Z}\times G)$ by

$$f(n,x)=j(n)g(x) \quad ((n,x)\in\mathbf{Z}\times G).$$

Then $Vf=j\otimes Sg$, so that $(Wf)^{(0)}=[T(j\otimes Sg)](0)=Sg\neq 0$ and, in fact, $W\neq 0$.

Now we consider the case $\mathbf{R}\times G$.

It follows from the above and from Lemma 1 and its proof that there exists a non-zero left invariant linear map $Y: C_{ru}(\mathbf{Z}\times G)\to L^2(\mathbf{Z}\times G)\cap$ $\cap\, C_0(\mathbf{Z}\times G)$ which is continuous both as a map into $L^2(\mathbf{Z}\times G)$ and as a map into $C_0(\mathbf{Z}\times G)$. By $\|Y\|$ we denote the norm of $Y: C_{ru}(\mathbf{Z}\times G)\to$ $\to L^2(\mathbf{Z}\times G)$.

For $f\in C_{ru}(\mathbf{R}\times G)$ and $s\in\mathbf{R}$ let f^s denote the restriction of $f_{(s,1)}$ to $\mathbf{Z}\times G$:

$$f^s(m,x)=f(m+s,x) \quad (m\in\mathbf{Z};\ x\in G).$$

If $s,t\in\mathbf{R}$ and $x,y\in G$, then $\|f_{(0,x)}{}^s-f_{(0,y)}{}^t\|\leqslant\|f_{(s,x)}-f_{(t,y)}\|$. As

$$Y: C_{ru}(\mathbf{Z}\times G)\to C_0(\mathbf{Z}\times G)$$

is continuous, for every $f\in C_{ru}(\mathbf{R}\times G)$ we can define a continuous bounded function Zf on $\mathbf{R}\times G$ by

$$(Zf)(s,x)=[Y(f^s)](0,x) \quad (s\in\mathbf{R};\ x\in G).$$

Then

$$(Zf)(s,x)=[Y(f_{(0,x)}{}^s)](0,1) \quad (s\in\mathbf{R};\ x\in G).$$

Clearly Z is linear. From the formula

$$f_{(t,xy)}{}^s=f_{(0,xy)}{}^{s+t} \quad (s,t\in\mathbf{R};\ x,y\in G)$$

it follows that $Z(f_{(t,y)})=(Zf)_{(t,y)}$ for all f, t, y. Thus Z is left invariant. It remains to prove that Z is non-zero and Z maps $C_{ru}(\mathbf{R}\times G)$ continuously into $L^2(\mathbf{R}\times G)$. For $s\in\mathbf{R}$, $m\in\mathbf{Z}$, $x\in G$ we have

$$(Zf)^s(m,x)=(Zf)(s+m,x)=[Y(f_{(0,x)}{}^{s+m})](0,1)=$$
$$=[Y((f^s)_{(m,x)})](0,1)=[Y(f^s)]_{(m,x)}(0,1)=[Y(f^s)](m,x).$$

Hence,

$$(Zf)^s=Y(f^s) \quad (s\in\mathbf{R}).$$

Consequently, $Z \neq 0$ and

$$\|Zf\|_2^2 = \sum_{n \in Z} \int_n^{n+1} \int_G |(Zf)(s, x)|^2 \, dx ds = \sum_n \int_0^1 \int_G |(Zf)(n+s, x)|^2 \, dx ds =$$

$$= \int_0^1 \sum_n \int_G |(Zf)^s(n, x)|^2 \, dx ds = \int_0^1 \|(Zf)^s\|^2 \, ds =$$

$$= \int_0^1 \|Y(f^s)\|^2 \, ds \leqslant \int_0^1 \|Y\|^2 \|f^s\|^2 \, ds \leqslant \int_0^1 \|Y\|^2 \|f\|^2 \, ds \leqslant \|Y\|^2 \|f\|^2.$$

Hence, $Zf \in L^2(\mathbb{R} \times G)$ and, in fact, $\|Zf\| \leqslant \|Y\| \, \|f\|$ for every f.

COROLLARY 11. *For every abelian locally compact group G there exists a non-zero left invariant continuous linear map $C_{ru}(G) \to L^2(G)$.*

PROOF. By [3; 24.30] we may assume G to be of the form $\mathbb{R}^n \times G_0$ where $n \in \{0, 1, \ldots\}$ while G_0 is a locally compact abelian group with a compact open subgroup. Being abelian, G_0 is amenable [3; 17.5]. Now apply Lemma 10 and Theorem 8.

Lemma 4 and the above corollary now yield our

MAIN THEOREM 12. *Let G be an abelian locally compact group. Then there exist non-zero multipliers $L^\infty(G) \to L^p(G)$ $(2 < p < \infty)$ and $L^\infty(G) \to$ $\to C_0(G)$.*

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